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The Performance Evaluation of Window Functions and Application to FIR Filter Design

Mridula Malhotra

Abstract — The paper is based upon the performance of various windows in designing FIR Filter. The cosine hyperbolic window has been used to design a better FIR Filter in terms of ripple ratio, side-lobe roll off ratio and main-lobe width with its advantages of no power series expansion in frequency domain hence need less hardware. The spectral characteristic of cosh window is compared with other windows like Kaiser Window, hamming window and their combinations. A modification is also introduced in cosh window to perform better in terms of ripple ratio as compared to Kaiser as well as cosh window.

Keywords- Cosh window, FIR filter design, Hamming window, Kaiser Window, Main-lobe width, Side-lobe roll-off ratio, Ultra spherical window, window function.

1. INTRODUCTION

Many Window functions are widely used in digital signal processing for various applications in signal analysis and estimation, digital filter design and speech processing [1]. In literature many windows have been proposed like ultra spherical window, Kaiser Window and hamming window with different specifications. But since they are suboptimal solutions, as there is a tradeoff between various factors and the best window depends upon the related application.

Kaiser window(also called I0-sinh window) [3] is a well known flexible window and widely used for the spectrum analysis and FIR filter design applications since it achieves a close approximation with the discrete prolate spheroidal functions that have the maximum energy concentration in the main-lobe. Because of the difficulty of computing the prolate function, a much simpler approximation using the zeroth-order modified Bessel function of the first kind was used which resulted in Kaiser window, defined as

$$wa(t) = \begin{cases} \dfrac{I_0[\alpha\sqrt{1-\left(\frac{t}{\tau}\right)^2}]}{I0[\alpha]} & |t| \le \tau \\ 0 & |t| > \tau \end{cases} \qquad (1)$$

The Fourier transform of Wa(t) is given by

$$wa(\omega)=\begin{cases} \dfrac{2\tau \, Sinh[\alpha\sqrt{(1-\left(\frac{\omega}{\omega_a}\right)^2}}{-I0[\alpha]\alpha\sqrt{(1-\left(\frac{\omega}{\omega_a}\right)^2}} & |\omega| \le \omega_a \\ \dfrac{2\tau \, Sinh[\alpha\sqrt{(\left(\frac{\omega}{\omega_a}\right)^2-1)}}{I0[\alpha]\alpha\sqrt{(\left(\frac{\omega}{\omega_a}\right)^2-1)}} & |\omega| > \omega_a \end{cases} \qquad (2)$$

Where $\alpha = \omega a \, \tau$.

By adjusting its two independent parameters α and τ, it can control the spectral parameters such as the main lobe width and the ripple ratio for various applications. The main-lobe width determines the ability to resolve adjacent spectral lines and the ratio determines leakage or interaction between spectral lines. This window gives design formula that accurately predicts the value of α and window duration 2τ required to achieve a prescribed transition region and approximation error for frequency selective filter design.

Side-lobe roll-off ratio is another spectral parameter and important for many applications. For beam forming applications, the higher side-lobe roll-off ratio means that it can reject the far end interferences better. The use of ultra spherical polynomials (Gegenbauer polynomials) for window designs introduces a degree of freedom relative to adjustable windows which allows one to achieve a variety of side-lobe patterns in the window's spectral representation and thus we can obtain desired side-lobe ratio. For filter design applications, it can reduce the far end attenuation for stop-band energy. Direct truncation of the infinite-duration impulse response of a filter leads to the well-known Gibbs' Phenomenon, which manifests itself in the form of large pass-band and stop-band ripples near transition bands. It is found that ultra spherical window yields lower order filters relative to designs obtained with the Kaiser and Dolph chebyshev windows. Alternatively, for a fixed filter length the ultra spherical window gives reduced pass band ripple and increased stop-band attenuation relative to Kaiser and Dolph chebyshev window. Through the use of window functions, the amplitude of the pass-band and stop band ripples can be reduced thereby improving the filter characteristics. using high side lobe falloff ratio combinational window provide better far end rejection of the stop-band energy. This feature helps to reduce the aliasing energy leak into a sub-band from that of the signal in the other sub band [6].

In this paper, study of window function based on the cosine hyperbolic function has been obtained to get better performance than Kaiser window. The cosh window is compared with Kaiser in terms of the window spectral parameters. Also, the application of the cosh window in the filter design is presented.

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2. INTRODUCTION TO THE COSH WINDOW

2.1. Spectral Properties of Windows

A window function, w(nT), with a length of N is a time domain function defined as nonzero for n≤|(N-1)/2|and zero for otherwise. Then the frequency spectrum of w(nT) can be found by

$$W(e^{jwT})=|A(w)|e^{j\theta(w)}=w(0)+2\sum_{n=1}^{(N-1)/2} w(nT)\cos wnT \quad (3)$$

where T is the sampling period. A typical window as a normalized amplitude spectrum in dB range as shown in Figure 1.

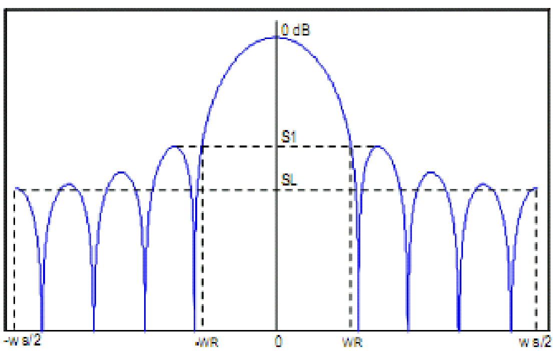


Figure1. A typical window's normalized amplitude spectrum.[1]

Normalized spectrum in Fig.1 is obtained by the following equation

$$|WN\ (ejwT)| = 20\log 10\ (|A(w)|\ /\ |A(W)|max) \quad (4)$$

The common spectral characteristic parameters to distinguish windows performance are the main-lobe width (wM), the ripple ratio (R) and the side-lobe roll-off ratio (S). From Fig. 1, these parameters can be defined as

wM = Main-lobe width =Twice the half mainlobe width 2wR

R = Maximum side-lobe amplitude in dB - Main-lobe Amplitude in dB =0 - S1

S = Maximum side-lobe amplitude in dB- Minimum side-lobe amplitude in dB = S1-SL

In the applications, smaller ripple ratio and narrower main-lobe width are required. But, this requirement is contradictory because, to reduce main-lobe width the window length should be increased which further increases ripple ratio. So there is a trade-off between these two parameters.

2.2. Kaiser Window

In discrete time domain, Kaiser window (or I0-sinh) is defined by [3]

$$wk(n)=\begin{cases} \dfrac{I_0\left(\alpha_k\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right)}{I_0(\alpha_K)} & |n| \le (N-1)/2 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Where αk is the adjustable parameter and I0(x) is the zeroth order modified Bessel function of the first kind which is described by a power series expansion as

$$I0(x) = 1+\sum_{k=1}^{\infty}\left[\frac{\left(\frac{x}{2}\right)^k}{k!}\right]^2 \quad (6)$$

C. Cosine Hyperbolic Window function [7]

The hyperbolic cosine of x is expressed as

$$\text{Cosh}(x)=(e^{x}+e^{-x})/2 \quad (7)$$

The functions cosh(x) and Io(x) are similar in shape and characteristic. Thus a new window can be represented as

$$wc(n)=\begin{cases} \dfrac{\cosh\left[\alpha_c\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right]}{\cosh[\alpha_c]} & |n| \le (N-1)/2 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

From equation (1) the cosh window for αc= 0 corresponds to the basic rectangular window. As for the case in Kaiser window, the exact spectrum for cosh window can be obtained from (1).

The frequency domain characteristics of cosh window shows that the main-lobe width increases and ripple ratio become smaller as we increase the value of αc, as shown in fig.3.

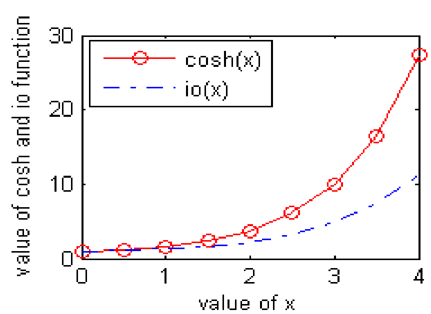


Figure 2. Comparison of the functions Io(x) and cosh(x)

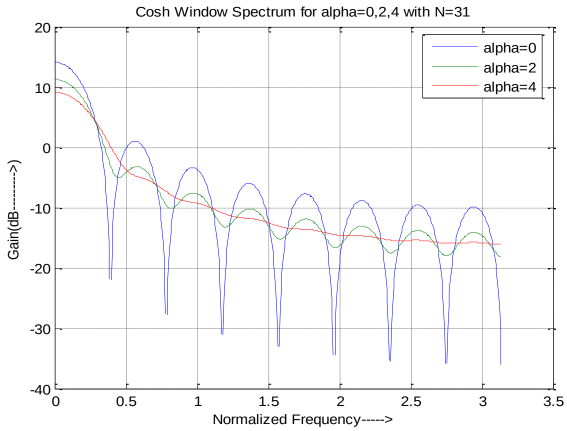


Figure 3.Cosh window spectrums for αc = 0, 2, and 8 with N=31

There is a relationship between the adjustable parameter α and the ripple ratio for the cosh window and shows that the ripple ratio remains almost constant for a change in the window length. An approximate relationship for αc in terms of R has been found by using the quadratic polynomial curve fitting method as[7]

$$\alpha=\begin{cases}0 & R>-13.26\\ -7.18*10^{-4}R^2-0.225R-2.519 & R\leq-13.26\end{cases} \quad (9)$$

Another relation can be found between the window length and the ripple ratio. To determine the window length for a given R and wR, the normalized width D=2wR(N-1) can be used [7]. By using quadratic polynomial curve fitting method, an approximate design relationship between D and R has been found as

$$D=\begin{cases}0 & R>-13.26\\ -7.84810^{-4}R^2-0.616R-1.81 & R\leq-13.26\end{cases} \quad (10)$$

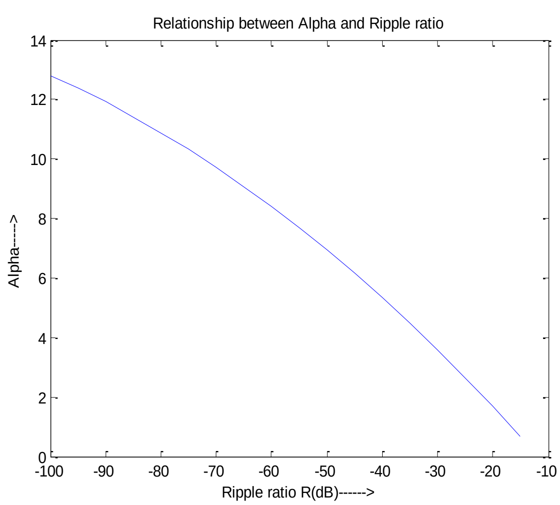


Fig.3 Relationship between ripple ratio(R) and parameter alpha

We can also predict a value of window length N as

$$N\geq\left(\frac{D}{2w_R}\right)+1 \quad (11)$$

Using (7) through (9), a cosh window can be designed for satisfying the given prescribed values of ripple ratio and main-lobe width. While increasing the side-lobe roll-off ratio it is observed that the normalized width also increases.

3. COMPARISIONS OF WINDOW SPECTRUM

3.1 Comparison with Kaiser Window

The comparison of cosh and Kaiser Window in terms of the ripple ratio suggests that Kaiser window gives lesser ripple ratio than cosh window, and the difference becomes larger as the normalized width increases.

On the other hand it is observed that cosh window gives better roll-off ratio than Kaiser, and the difference becomes larger as the normalized width increases.

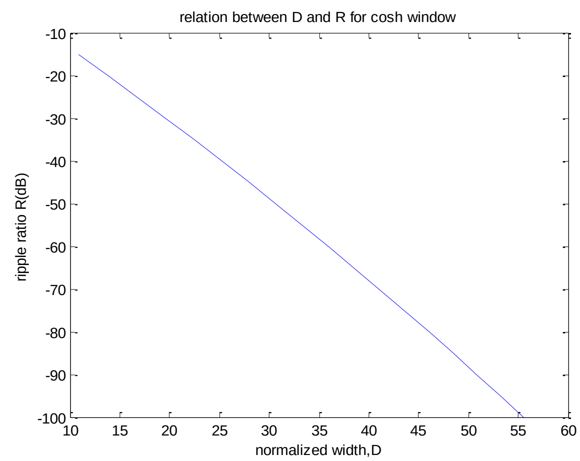
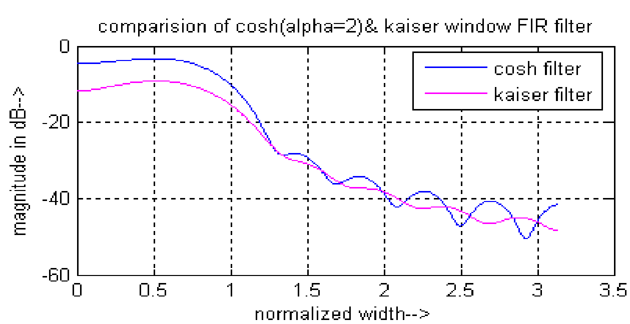


Fig.4 Relationship between ripple ratio(R) and normalized width(D).



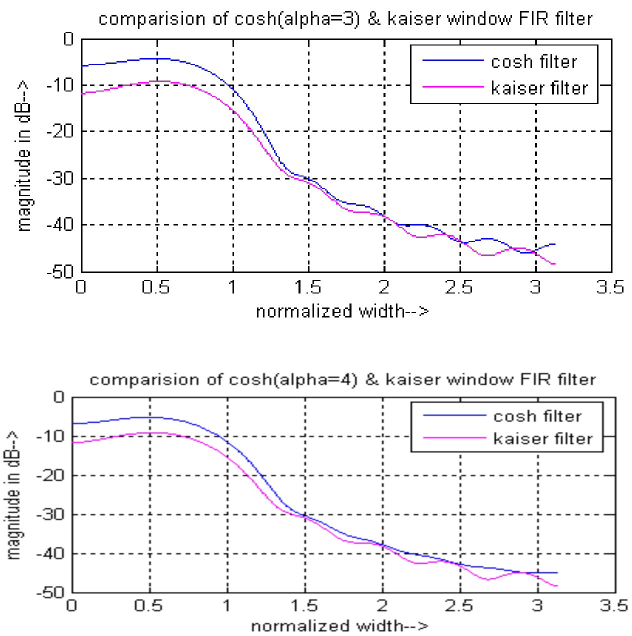


Figure 7. Comparison of Kaiser window and cosh window function for N=31.

Table1. Comparison of combination of various window function.

WINDOW	N	ALPHA	W_R	R(dB)	S(dB)
KAISER	15	3.9754	1.638	-28.22	3.33
COSH	15	2	1.663	-26	2.99
COSH	15	3	1.786	-27.8	2.44
COSH	15	4	1.859	-28.1	2.05
KAISER	31	3.9754	1.313	-32.3	13.68
COSH	31	2	1.331	-28.64	13.03
COSH	31	3	1.374	-30.95	12.06
COSH	31	4	1.387	-32.3	11.25

3.2. Comparison with the Combinational Windows Including Hamming Window

To provide a better window spectral characteristics windows can be used in the combination. From the previous section, it was observed that cosh window has a worse ripple ratio than Kaiser Window. To improve its ripple ratio, it can be combined with Hamming window [1].

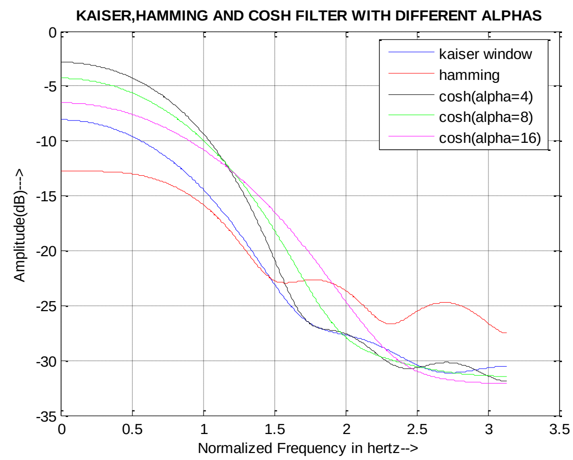


Figure 8 Comparison of various window function.

By comparing cosh, Kaiser and the combinational windows including Hamming window for a fixed window length and main-lobe width, it can be observed that the maximum side-lobe amplitude for the ripple ratio occurs in the first side-lobe but that it lies in the third side-lobe for the combination of Kaiser and Hamming windows.
The combination of cosh and Hamming window has the smallest ripple ratio. This shows that cosh window can perform better results than Kaiser window in terms of the ripple ratio if it can be combined with a suitable window such as Hamming window.

4. WINDOW FUNCTION FOR FIR FILTER DESIGN

4.1. Filter Design Using Window Method

While designing a filter using Fourier series, the pass-band and stop-band oscillations are observed due to slow convergence in the Fourier series, which in turn is caused by the discontinuity at ωc(cut-off frequency). These are known as Gibbs' oscillations.
As N is increased, the frequency of these oscillations is seen to increase, and at both low and high frequencies their amplitude is decreased. Also the transition between pass-band and stop-band becomes steeper. However, the amplitudes of the pass-band and stop-band ripples closest to the pass-band edge remain virtually unchanged. Consequently, the quality of the filter obtained is not very good and ways must be found for the reduction of Gibbs' oscillation.

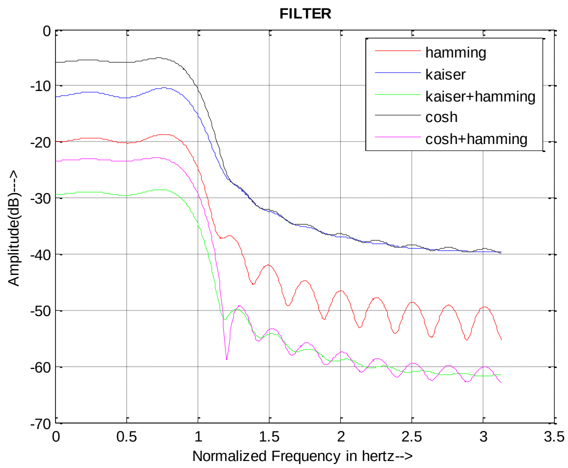


Figure 9 Comparison of combination of various window function.

When a Fourier series is truncated, it will exhibit certain oscillations known as Gibbs' oscillations. Gibbs' oscillations are most pronounced near discontinuities and due to the slow convergence of the Fourier series. The amplitude of Gibbs oscillations tends to be independent of the number of terms retained in the Fourier series. Fourier series method with windowing is the most straight forward technique to design FIR filters and involves a minimal amount of computation compared to the optimization methods.

Thus a window truncates and smooths the infinite duration impulse response of the filter by using fourier series. The impulse response is obtained as

$$hnc(nT)= w(nT)\, hid(nT) \quad (12)$$

Table 2. Comparison of combination of various window function.

WINDOW FILTER	N	wR	α	R	S
COSH+ HAMMING	51	1.331	-	-54.41	3.35
KAISER+ HAMMING	51	1.276	-	-53.24	7.74
KAISER	51	1.197	4.312	-28.82	10.8
COSH	51	1.233	4	-28.51	10.54
HAMMING	51	1.16	-	-36.78	12.57

where hid(nT) is the infinite duration impulse response of the ideal filter. The window function can be of any type depending upon the application [1,2,3]. For a low pass filter it can be found as [1]

$$hid(nT)=\begin{cases} w_{ct}T\,\pi & \text{for } n = 0 \\ \sin\ w_{ct}nT/n\pi & \text{for } w_{ct} < |w| \leq w_s/2 \end{cases} \quad (13)$$

where wct= cut off frequency, ws = sampling frequency
A casual filter can be obtained as

$$h(Nt)=hnc[(n-(N-1)/2)T] \quad \text{for } 0\leq n\leq N-1 \quad (14)$$

By using window method the ripples in pass-band and stop-band regions of the filters are approximately equal [4].

4.2. Design Equations of Cosh Window for filter design

The window parameters and filter parameter must be matched to satisfy the design specification. As the window parameter increases the minimum stop-band attenuation also increases, thus by using the quadratic polynomial curve fitting method, an approximate design equation has been found[7] for the attenuation in range 21 ≤ As ≤100dB, as

$$\alpha c = -2.582*10^{-4}A_s^{\,2} + 0.16A - 2.924 \quad (15)$$

Using the quadratic polynomial curve fitting method, an approximate expression for D has been found for 21≤ As≤100dB

$$D= -3.52*10^{-5}A_S^2 + 0.07862A_S - 0.07056 \quad (16)$$

Where D is normalized width given as,

$$D=\Delta w(N-1)/W_S \quad (17)$$

And Δ w is the transition bandwidth.
By using (15), the minimum filter length required for satisfying a given As and Δw is given by

$$N\geq (DWs/\Delta w) +1 \quad (18)$$

As a result, using the filter design equations given in (15), (16) and (18) a cosh window has been designed to satisfy the prescribed filter characteristic given in terms of As and Δw. Kaiser window gives better minimum stop-band attenuation characteristic than a filter designed by cosh window.
Another comparison with Kaiser window is the far end stop-band attenuation, which also gives the maximum stop-band attenuation, is taken as a figure of merit.
The attenuation of the far end in stop-band is important for many applications [6].It is observed that as the transition width increases, the filters designed by cosh window

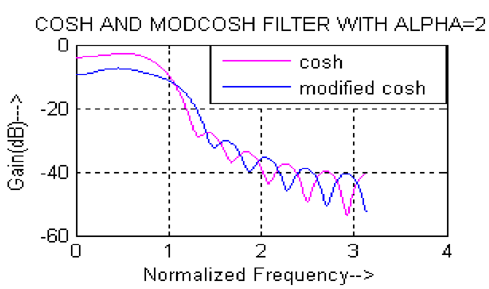
performs better far end attenuation as compared to the filters designed by Kaiser window.

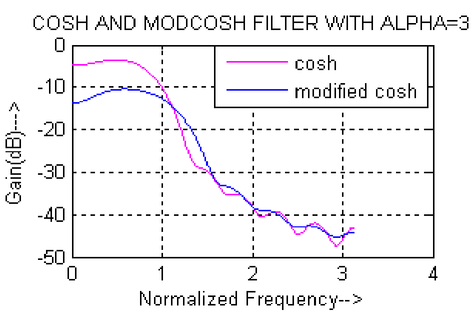
5. MODIFICATION IN COSH WINOW

It is seen that the combination of cosh window with hamming window can somehow reduces the ripple ratio. To reduce the complexity and for a better response we can introduce a multiplicative factor, p, similar to the adjusting parameter α, which gives the new window function

$$W(n)=\cosh(\alpha\sqrt{\{1-((2n/N-1)^2)*p\}}\;/\;\cosh(\alpha) \qquad (19)$$

By introducing a third parameter (p) in the window function a better window function can be obtained for FIR filter design where higher main-lobe width and smaller ripple ratio is important. It also leads to better side-lobe roll off ratio. Thus it gives a better response over cosh window, Kaiser window and their combination with hamming window as shown in Fig.9 and Fig.10. The parameters are compared in Table 2 and Table 3.





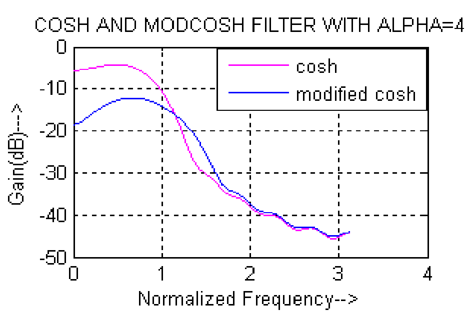


Figure 10 Comparision of Modified cosh Filter and cosh Filter
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Table 3. Comparison of cosh filter and modified cosh filter(N=31)

WINDOW	P	N	α	WR	R(dB)	S(dB)
COSH	-	31	2	1.319	-27.73	13.15
MODIFIED COSH	2	31	2	1.442	-31.03	11.94
	4	31	2	1.485	-31.24	10.43
	6	31	2	1.54	-32.39	10.31
COSH		31	3	1.356	-30.07	12.06
MODIFIED COSH	2	31	3	1.485	-32.01	10.88
	4	31	3	1.614	-33.98	8.84
	6	31	3	1.694	-34.12	8.81
COSH		31	4	1.436	-31.16	11.92
MODIFIED COSH	2	31	4	1.546	-33.34	10.72
	4	31	4	1.718	-35.42	7.45
	6	31	4	1.871	-36.63	6.94

6. CONCLUSION

In this paper, cosh window is analyzed and compared with various window functions and their combination. Cosh window has been used to design a better FIR Filter in terms of ripple ratio, side-lobe roll off ratio and main-lobe width. But Kaiser window gives better ripple ratio as compared to Kaiser window. This drawback can be overcome by the combination of hamming window with cosh window. Further the complexity can be reduced by introducing a modified window having a new adjusting parameter similar to α that also increases the mainlobe width and reduces the sidelobe ratio and ripple ratio.

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